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A random walk between short and long memory

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Outline

Sequential Empirical Process

- $(X_n)_{n\in\mathbb{N}}$ stationary sequence of real valued random variables
- sequential empirical process: two parameter process

$$(W_n(s,t))_{s,t\in[0,1]}$$

with

$$W_n(s,t) := \sum_{i=1}^{[nt]} \left(\mathbb{1}_{\{X_i\le s\}} - F(s)\right)$$

- fixed s: partial sum process
- fixed t: empirical process
- without loss of generality: $F(s) = s$ for $s \in [0,1]$

First Result on Sequential Empirical Process

Theorem (Müller, 1970)

Let $(X_n)_{n\in\mathbb{N}}$ be iid, then

$$\left(\frac{1}{\sqrt{n}}W_n(s,t)\right)_{s,t\in[0,1]} \Rightarrow (K(s,t))_{s,t\in[0,1]}$$

where K is a centered Gaussian process with
$\mathrm{Cov}\,(K(s,t),K(s',t')) = \min\{t,t'\}(\min\{s,s'\} - ss')$

limit process called Kiefer-Müller-process

extensions to short range dependence:

e.g. Berkes, Philipp (1977): approximating functionals of strongly mixing sequences

Properties of Kiefer-Müller-process

- for fixed s: $(K(s,t))_{t\in[0,1]}$ Brownian motion
- for fixed t: $(K(s,t))_{s\in[0,1]}$ Brownian bridge
- self-similarity with exponent $\frac{1}{2}$:
  $(K(s,at))_{s,t\in[0,1]}$ and $(a^{1/2}K(s,t))_{s,t\in[0,1]}$ have same distribution
- rough paths: $(K(s,t))_{s,t\in[0,1]}$ not Lipschitz-continuous,
  γ-Hölder-continuous for any $\gamma \in (0,\frac{1}{2})$

Long Range Dependent Gaussian Sequences

$X_i = G(\xi_i)$, where $(\xi_n)_{n\in\mathbb{N}}$ stationary, Gaussian process with $E[\xi_i] = 0$, $\mathrm{Var}[\xi_i] = 1$ and

$$r(k) := \mathrm{Cov}(\xi_1, \xi_{1+k}) = k^{-D} L(k)$$

$D \in (0, 1)$ and slowly varying function $L(k)$

Hermite polynomials: orthogonal with respect to Gaussian measure
$H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$,...

Hermite rank:

$$r := \min \left\{ k \middle| E\left[G(\xi_1) H_k(\xi_1)\right] \neq 0 \right\}$$

Sequential Empirical Process under Long Memory

Hermite rank: $m := \min\left\{k \middle| \exists s : \; E\left[(\mathbb{1}_{\{X_1 \leq s\}} - F(s))H_k(\xi_1)\right] \neq 0\right\}$

$$W_n(s,t) := \sum_{i=1}^{[nt]} \left(\mathbb{1}_{\{X_i \leq s\}} - F(s)\right)$$

Theorem (Dehling, Taqqu, 1989)

If $D \in (0, \frac{1}{m})$

$$\left(n^{\frac{mD-2}{2}} L^{-1/2}(n) W_n(s,t)\right)_{s,t \in [0,1]} \Rightarrow \left(C_{m,D} J_m(s) Z_m(t)\right)_{s,t \in [0,1]}$$

with $J_m(s) = E\left[(\mathbb{1}_{\{X_1 \leq s\}} - F(s))H_m(\xi_1)\right]$ and Z_m: Hermite process of order m

Ho and Hsing (1996): linear processes with long memory

Properties of Dehling-Taqqu-Type Limit

- **semi-degenerate limit:** $(G(s,t))_{s\in[0,1]}$ for fixed t deterministic function with random factor
- for fixed s: $(G(s,t))_{t\in[0,1]}$ Hermite process
- self-similarity with exponent $\frac{2-mD}{2}$
- not Gaussian, if $mD < 1$, $m \geq 2$
- rough paths in t direction: $(G(s,t))_{t\in[0,1]}$ not Lipschitz-continuous, γ-Hölder-continuous for any $\gamma < \frac{2-mD}{2}$
- smooth paths in s direction: $(G(s,t))_{s\in[0,1]}$ Lipschitz-continuous if F Lipschitz continuous

Do all processes with long memory have a semi-degenerate limit of the sequential empirical process?

Dynamical System with Mild Long Memory

- stationary process, $X_{i+1} = T(X_i)$ with

$$T(x) = \begin{cases} x(1+\sqrt{2x}) & \text{for } x \in [0, \frac{1}{2}) \\ 2x - 1 & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

- $\mathrm{Cov}\,(X_i, X_{i+k}) \approx C\frac{1}{k}$

Theorem (Dedecker, Dehling, Taqqu, 2015)

$$\left(\frac{1}{\sqrt{n \log n}} W_n(s,1)\right)_{s\in[0,1]} \Rightarrow (g(s)Z)_{s,t\in[0,1]}$$

Z Gaussian

Definition of Random Walk in Random Scenery

random walk: $(X_n)_{n\in\mathbb{N}}$ iid $\mathbb{Z}$-valued random variables in the normal domain of attraction of an α-stable law F_α with $0 < \alpha \leq 2$

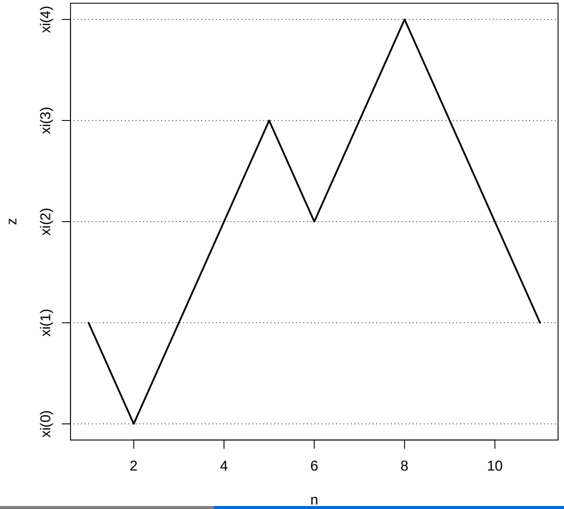
$$S_n := \sum_{m=1}^{n} X_m$$

scenery: $(\xi(i))_{i\in\mathbb{Z}}$ iid $\mathbb{R}$-valued random variables in the normal domain of attraction of an β-stable law F_β with $0 < \beta \leq 2$

random walk in random scenery:

$$(\xi(S_i))_{i\in\mathbb{N}}$$

Definition of Random Walk in Random Scenery II



random walk:
$S_1 = 1$, $S_2 = 0$,
$S_3 = 1$, $S_4 = 2$,
$S_5 = 3$, ...

random walk
in random scenery:
$\xi(S_1) = \xi(1)$,
$\xi(S_2) = \xi(0)$,
$\xi(S_3) = \xi(1)$,
$\xi(S_4) = \xi(2)$,
$\xi(S_5) = \xi(3)$, ...

Properties of Random Walk in Random Scenery

- ergodic
- not absolutely regular (β-mixing)
- long range dependent
  if S_n simple random walk, $\beta = 2$, then

$$\mathrm{Cov}(\xi(S_i), \xi(S_{i+2k})) = P(S_i = S_{i+2k})\,\mathrm{Cov}(\xi(1), \xi(1)) \approx \frac{1}{\sqrt{k}}\frac{1}{\sqrt{\pi}}\mathrm{Var}(\xi(1))$$

- heavy tails, if $(\xi(i))_{i\in\mathbb{Z}}$ has heavy tails

Limit Theorem for Random Walks in Random Scenery

Theorem (Kesten, Spitzer, 1979)

For $1 < \alpha \leq 2$

$$\left(n^{-1+\frac{1}{\alpha}-\frac{1}{\alpha\beta}}\sum_{i=1}^{[nt]}\xi(S_i)\right)_{t\in[0,1]} \Rightarrow (\Delta_t)_{t\in[0,1]}$$

properties of Δ

- self-similar with exponent $1-\frac{1}{\alpha}+\frac{1}{\alpha\beta}$
- non Gaussian even if $\beta = 2$ (scenery has Gaussian limit)
- continuous even if $\beta < 2$ (scenery has limit with jumps)

Definition of Limit Process

$$\sum_{i=1}^{n} \xi(S_i) = \sum_{x\in\mathbb{Z}} N_n(x)\xi(x)$$

with occupation times

$$N_n(x) = \sum_{i=1}^{n} \mathbb{1}_{\{S_i=x\}}$$

$$\Delta_t = \int_{-\infty}^{\infty} L_t(x)dZ(x)$$

with Z: two-sided β-stable process
L: local time of an α-stable process $S^\star$, i.e.

$$\int_0^t \mathbb{1}_{[a,b)}(S_s^\star)ds = \int_a^b L_t(x)dx$$

Limit Theorem for Sequential Empirical Process

- $S_n := \sum_{m=1}^{n} X_m$ with α-stable limit, $1 < \alpha \leq 2$
- $W_n(s,t) := \sum_{i=1}^{[nt]} \left(\mathbb{1}_{\{\xi(S_i)\leq s\}} - F(s)\right)$
- without loss of generality: $F(s) = P(\xi_i \leq s) = s$

Theorem (Wendler, 2016)

$$n^{-1+\frac{1}{2\alpha}} W_n \Rightarrow W$$

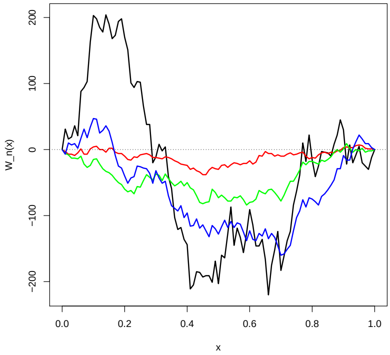
in the space $D\left([0,1]^2\right)$

with

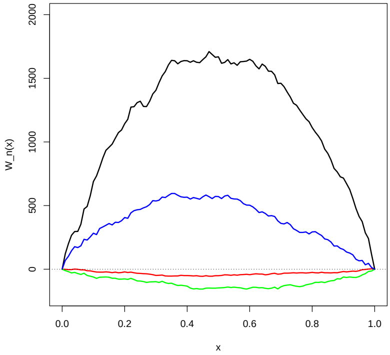
- $W(s,t) := \int_{\mathbb{R}} L_t(x) dK(s,x)$
- $(K(s,t))_{s,t\in[0,1]}$: Kiefer-Müller process
- local time L

Comparison: space direction

- sequential empirical process $W_n(\cdot, 1)$
- sample size $n = 1000$, $n = 4000$, $n = 16000$, $n = 64000$



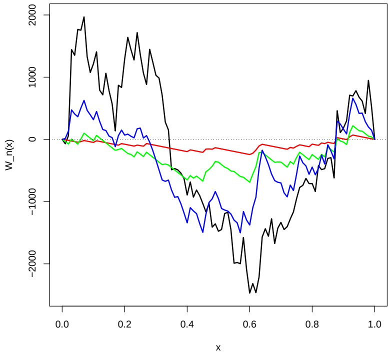
independent



long range dependent Gaussian

Comparison: space direction

- sequential empirical process $W_n(\cdot, 1)$
- sample size $n = 1000$, $n = 4000$, $n = 16000$, $n = 64000$

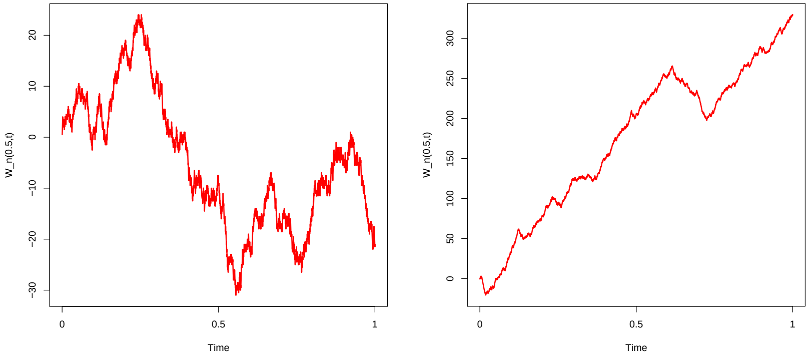


random walk in random scenery: nearest neighbour random walk

rough paths (like independent) non degenerate

Comparison: time direction

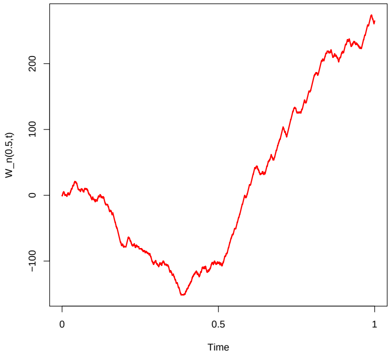
- sequential empirical process $W_n(0.5, \cdot)$
- sample size $n = 4000$



independent | long range dependent Gaussian

Comparison: time direction

- sequential empirical process $W_n(0.5, \cdot)$
- sample size $n = 4000$



random walk in random scenery: nearest neighbour random walk

self-similarity (like long range dependent)

Continuity of Limit Process

nonuniform Kolmogorov-Chentsov theorem

Lemma

$(\mathbf{X}_t)_{t\in[0,1]^d}$ stochastic process, for some $m \geq 1$, $c_1, \dots, c_d, \beta_1, \dots, \beta_d$:

$$E\left[|\mathbf{X}_t - \mathbf{X}_s|^m\right] \leq \sum_{i=1}^{d} c_i \left|t_i - s_i\right|^{d+\beta_i}.$$

for all $t = (t_1, \dots, t_d)$, $s = (s_1, \dots, s_d)$, then for $\gamma_1, \dots, \gamma_d$ with $\gamma_i < \frac{\beta_i}{m}$ there exists a continuous modification $\tilde{\mathbf{X}}$ with

$$\left|\tilde{\mathbf{X}}_t - \tilde{\mathbf{X}}_s\right| \leq C_{\gamma_1,\dots,\gamma_d} \sum_{i=1}^{d} \left|t_i - s_i\right|^{\gamma_i}.$$

Comparision: Summary

	independent	random walk in random scenery	lrd Gaussian
norming for convergence	$n^{-1/2}$	$n^{-1+\frac{1}{2\alpha}}$	$n^{-1+\frac{mD}{2}}$
exponent of self similarity	$1/2$	$1-\frac{1}{2\alpha} > 1/2$	$\frac{2-mD}{2} > 1/2$
marginal distr.	Gaussian	not Gaussian: Gaussian mixture	not Gaussian if $m > 1$
degeneracy	no	no	yes
γ_t-Hölder-cont. t direction	$\gamma_t < 1/2$	$\gamma_t > 1/2$	$\gamma_t > 1/2$
γ_s-Hölder-cont. s direction	$\gamma_s < 1/2$	$\gamma_s < 1/2$	$\gamma_s = 1$

Transient Case

transient: $(X_n)_{n \in \mathbb{N}}$ iid, normal domain of attraction of α-stable law with $\alpha < 1$

$$S_n := \sum_{m=1}^{n} X_m$$

Theorem (Guillotin-Plantard, Pène, Wendler, 2017+)

$$\frac{1}{\sqrt{n}} W_n \Rightarrow CK$$

in the space $D\left([0,1]^2\right)$

with

- $(K(s,t))_{s,t \in [0,1]}$: Kiefer-Müller process
- limit not degenerate

0-Recurrecnt Case

$(X_n)_{n\in\mathbb{N}}$ iid, normal domain of attraction of α-stable law with $\alpha = 1$

Theorem (Guillotin-Plantard, Pène, Wendler, 2017+)

$$\frac{1}{\sqrt{n\log n}} W_n \Rightarrow CK$$

in the space $D\left([0,1]^2\right)$

with $(K(s,t))_{s,t\in[0,1]}$: Kiefer-Müller process

other 0-recurrent case: $(S_n)_{n\in\mathbb{N}}$ simple random walk on $\mathbb{Z}^2$, $(\xi(i))_{i\in\mathbb{Z}^2}$ iid, random walk in random scenery $(\xi(S_n))_{n\in\mathbb{N}}$

Conclusion

random walk in random scenery:

- functional non-central limit theorem for sequential empirical process
- recurrent case: limit shares some properties with independence, some properties with long range dependence
- transient/ 0-recurrent case: limit like under independence

Thank you for your attention!

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