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The empirical process of a random walk in random scenery

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Outline

Introduction
- Sequential Empirical Process
- Random Walks in Random Scenery

Main Result
- Limit Theorem
- Application to U-Statistics

Empirical Distribution Function

- $(X_n)_{n\in\mathbb{N}}$ stationary sequence of real valued random variables
- distribution function: $F : \mathbb{R} \to [0,1]$ with $F(s) := P(X_i \leq s)$

empirical distribution function: $F_n : \mathbb{R} \to [0,1]$

- $F_n(s) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X_i \leq t}$
- natural estimator for F

Theorem (Glivenko-Cantelli)

Let $(X_n)_{n\in\mathbb{N}}$ be ergodic, then

$$\sup_{s\in\mathbb{R}} |F(s) - F_n(s)| \to 0$$

almost surely.

Sequential Empirical Process

- $(X_n)_{n\in\mathbb{N}}$ stationary sequence of real valued random variables
- sequential empirical process: two parameter process

$$(W_n(s,t))_{s,t\in[0,1]}$$

with

$$W_n(s,t) := \sum_{i=1}^{[nt]} \left(\mathbb{1}_{\{X_i \le s\}} - F(s)\right)$$

- fixed s: partial sum process
- fixed t: empirical process
- without loss of generality: $F(s) = s$ for $s \in [0,1]$

First Result on Sequential Empirical Process

Theorem (Müller, 1970)

Let $(X_n)_{n\in\mathbb{N}}$ *be iid, then*

$$\left(\frac{1}{\sqrt{n}} W_n(s,t)\right)_{s,t\in[0,1]} \Rightarrow (K(s,t))_{s,t\in[0,1]}$$

where K *is a centered Gaussian process with*
$\mathrm{Cov}\,(K(s,t), K(s',t')) = \min\{t,t'\}(\min(s,s') - ss')$

limit process called Kiefer-Müller-process

extensions to short range dependence:

e.g. Berkes, Philipp (1977): approximating functionals of strongly mixing sequences

Sketch of Proof

- for fixed s, t:

$$W_n(s,t) + E\left[W_n(s,t)\right] = \sum_{i=1}^{[nt]} \mathbb{1}_{\{X_i \le s\}}$$

  binomial($[nt]$,s) distributed
- finite-dimensional convergence: De Moivre-Laplace theorem
- tightness: exponential inequality
- note: Müller used $C_{D[0,1]}([0,1])$, but proof in $D([0,1]^2)$ possible

Properties of Kiefer-Müller-process

- for fixed s: $(K(s,t))_{t\in[0,1]}$ Brownian motion
- for fixed t: $(K(s,t))_{s\in[0,1]}$ Brownian bridge
- self-similarity with exponent $\frac{1}{2}$:
  $(K(s,at))_{s,t\in[0,1]}$ and $(a^{1/2}K(s,t))_{s,t\in[0,1]}$ have same distribution
- rough paths: $(K(s,t))_{s,t\in[0,1]}$ not Lipschitz-continuous,
  γ-Hölder-continuous for any $\gamma\in(0,\frac{1}{2})$

Long Range Dependent Gaussian Sequences

$X_i = G(\xi_i)$, where $(\xi_n)_{n\in\mathbb{N}}$ stationary, Gaussian process with $E[\xi_i] = 0$, $\text{Var}[\xi_i] = 1$ and

$$r(k) := \text{Cov}(\xi_1, \xi_{1+k}) = k^{-D}L(k)$$

$D \in (0, 1)$ and slowly varying function $L(k)$

Hermite polynomials: orthogonal with respect to Gaussian measure
$H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$,...

Hermite rank:

$$r := \min\left\{k \middle| E\left[G(\xi_1)H_k(\xi_1)\right] \neq 0\right\}$$

Hermite Expansion

L_2-approximation of $G(\xi_i)$

$$G(\xi_i) \stackrel{L_2}{=} \sum_{k=r}^{\infty} \frac{c_k}{k!} H_k(\xi_i)$$

$$\Rightarrow \quad \sum_{i=1}^{n} G(\xi_i) \stackrel{L_2}{=} \sum_{k=r}^{\infty} \frac{c_k}{k!} \left(\sum_{i=1}^{n} H_k(\xi_i) \right)$$

with

$$c_k = E\left[G(\xi_1) H_k(\xi_1)\right]$$

and

$$\sum_{k=r}^{\infty} \frac{c_k^2}{k!} < \infty$$

Reduction Principle

- correlation of Hermite polynomials:

$$\mathrm{Cov}\left(H_k(\xi_1), H_k(\xi_{1+i})\right) = k!\left(\mathrm{Cov}\left(\xi_1, \xi_{1+i}\right)\right)^k$$

- variance of sums of Hermite polynomials:

$$\mathrm{Var}\left[\left(\sum_{i=1}^{n} H_k(\xi_i)\right)\right] \approx \begin{cases} \beta_k n^{2-kD} L^k(n) & \text{if } kD < 1 \\ Ck!n & \text{if } kD > 1 \end{cases}$$

- sum of higher order Hermite polynomials: lower order of variance
- reduction principle: limit behaviour of $\sum_{i=1}^{n} G(\xi_i)$ dominated by $\frac{c_k}{k!}\left(\sum_{i=1}^{n} H_r(\xi_i)\right)$

Sequential Empirical Process under Long Memory

Hermite rank: $m := \min\left\{k \middle| \exists s: \ E\left[(\mathbb{1}_{\{X_1 \le s\}} - F(s))H_k(\xi_1)\right] \neq 0\right\}$

$$W_n(s,t) := \sum_{i=1}^{[nt]} \left(\mathbb{1}_{\{X_i \le s\}} - F(s)\right)$$

Theorem (Dehling, Taqqu, 1989)

If $D \in (0, \frac{1}{m})$

$$\left(n^{\frac{mD-2}{2}} L^{-1/2}(n) W_n(s,t)\right)_{s,t\in[0,1]} \Rightarrow \left(C_{m,D} J_m(s) Z_m(t)\right)_{s,t\in[0,1]}$$

with $J_m(s) = E\left[(\mathbb{1}_{\{X_1 \le s\}} - F(s))H_m(\xi_1)\right]$ and Z_m: Hermite process of order m

Ho and Hsing (1996): linear processes with long memory

Idea of Proof

Lemma (Uniform Reduction Principle)

$$\sup_{s,t\in[0,1]} \left| n^{\frac{mD-2}{2}} L^{-1/2}(n) \sum_{i=1}^{[nt]} \sum_{k=m+1}^{\infty} J_k(s) H_k(\xi_i) \right| \Rightarrow 0$$

$$\left(\frac{n^{\frac{mD-2}{2}}}{L^{m/2}(n)} W_n(s,t) \right)_{s,t\in[0,1]} \overset{L_2}{\approx} \left(\frac{n^{\frac{mD-2}{2}}}{L^{m/2}(n)} \sum_{i=1}^{[nt]} \frac{J_m(s)}{m!} H_m(\xi_i) \right)_{s,t\in[0,1]}$$

$$\Rightarrow \left(C_{m,D} J_m(s) Z_m(t) \right)_{s,t\in[0,1]}$$

Properties of Dehling-Taqqu-Type Limit

- for fixed s: $(G(s,t))_{t\in[0,1]}$ Hermite process
- for fixed t: $(G(s,t))_{s\in[0,1]}$ deterministic function with random factor (semi-degenerate limit)
- self-similarity with exponent $\frac{2-mD}{2}$
- not Gaussian, if $mD < 1$, $m \geq 2$
- rough paths in t direction: $(G(s,t))_{t\in[0,1]}$ not Lipschitz-continuous, γ-Hölder-continuous for any $\gamma < \frac{2-mD}{2}$
- smooth paths in s direction: $(G(s,t))_{s\in[0,1]}$ Lipschitz-continuous if F Lipschitz continuous

Do all processes with long memory have a Dehling-Taqqu-type limit of the sequential empirical process?

Definition of Random Walk in Random Scenery

random walk: $(X_n)_{n\in\mathbb{N}}$ iid $\mathbb{Z}$-valued random variables in the normal domain of attraction of an α-stable law F_α with $0 < \alpha \leq 2$

$$S_n := \sum_{m=1}^{n} X_m$$

scenery: $(\xi(i))_{i\in\mathbb{Z}}$ iid $\mathbb{R}$-valued random variables in the normal domain of attraction of an β-stable law F_β with $0 < \beta \leq 2$

random walk in random scenery:

$$(\xi(S_i))_{i\in\mathbb{N}}$$

Properties of Random Walk in Random Scenery

- ergodic
- not absolutely regular (β-mixing)
- long range dependent
  if S_n simple random walk, $\beta = 2$, then

$$\mathrm{Cov}(\xi(S_i), \xi(S_{i+2k})) = P(S_i = S_{i+2k})\,\mathrm{Cov}(\xi(1), \xi(1)) \approx \frac{1}{\sqrt{k}}\frac{1}{\sqrt{\pi}}\mathrm{Var}(\xi(1))$$

- heavy tails, if $(\xi(i))_{i\in\mathbb{Z}}$ has heavy tails

Limit Theorem for Partial Sums

Theorem (Kesten, Spitzer, 1979)

For $1 < \alpha \leq 2$

$$\left(n^{-1+\frac{1}{\alpha}-\frac{1}{\alpha\beta}} \sum_{i=1}^{[nt]} \xi(S_i) \right)_{t\in[0,1]} \Rightarrow (\Delta_t)_{t\in[0,1]}$$

properties of Δ

- self-similar with exponent $1 - \frac{1}{\alpha} + \frac{1}{\alpha\beta}$
- non Gaussian even if $\beta = 2$ (scenery has Gaussian limit)
- continuous even if $\beta < 2$ (scenery has limit with jumps)

Definition of Limit Process

$$\sum_{i=1}^{n} \xi(S_i) = \sum_{x \in \mathbb{Z}} N_n(x)\xi(x)$$

with occupation times

$$N_n(x) = \sum_{i=1}^{n} \mathbb{1}_{\{S_i = x\}}$$

$$\Delta_t = \int_{-\infty}^{\infty} L_t(x) dZ(x)$$

with Z: two-sided β-stable process
L: local time of an α-stable process $S^\star$, i.e.

$$\int_0^t \mathbb{1}_{[a,b)}(S_s^\star) ds = \int_a^b L_t(x) dx$$

Sequential Empirical Process

$(X_n)_{n\in\mathbb{N}}$ iid, in the normal domain of attraction of an α-stable law with $1 < \alpha \le 2$

$$S_n := \sum_{m=1}^{n} X_m$$

sequential empirical process:

$$W_n(s,t) := \sum_{i=1}^{[nt]} \left(1_{\{\xi(S_i)\le s\}} - F(s)\right)$$

no reduction principle: if S_n simple random walk, then for all $p \in \mathbb{N}$

$$\mathrm{Cov}(H_p(\xi(S_i)), H_p(\xi(S_{i+2k}))) \approx C_p \frac{1}{\sqrt{k}}$$

Limit Theorem for Sequential Empirical Process

without loss of generality: ξ_i uniformly distributed on $[0,1]$

Theorem (W., 2016)

$$n^{-1+\frac{1}{2\alpha}} W_n \Rightarrow W$$

in the space $D\left([0,1]^2\right)$

with

- $W(s,t) := \int_{\mathbb{R}} L_t(x) dK(s,x)$
- $(K(s,t))_{s,t\in[0,1]}$: Kiefer-Müller process
- local time L

Sketch of Proof I

- occupations times

$$N_n(x) := \sum_{i=1}^{n} \mathbb{1}_{\{S_i = x\}}.$$

- L: local time

$$\int_0^t \mathbb{1}_{[a,b)}(S_s^\star)ds = \int_a^b L_t(x)dx$$

Lemma

for $k \in \mathbb{N}$, $t_1, \ldots, t_k \in [0,1]$

$$\left(n^{-2+\frac{1}{\alpha}} \sum_{x\in\mathbb{Z}} N_{[nt_i]}(x) N_{[nt_j]}(x)\right)_{1\le i,j\le k} \Rightarrow \left(\int L_{t_i}(x) L_{t_j}(x) dx\right)_{1\le i,j\le k}$$

Sketch of Proof II

- finite dimensional convergence: use characteristic function and Cramér-Wold theorem
-

$$n^{-1+\frac{1}{2\alpha}} \sum_{j=1}^{k} \theta_j \sum_{i=1}^{[nt_j]} \left(\mathbb{1}_{\{Y_i \le s_j\}} - s_j \right)$$

$$= n^{-1+\frac{1}{2\alpha}} \sum_{j=1}^{k} \theta_j \sum_{x \in \mathbb{Z}} N_{[nt_j]}(x) \left(\mathbb{1}_{\{\xi_x \le s_j\}} - s_j \right)$$

$$= n^{-1+\frac{1}{2\alpha}} \sum_{j=1}^{k} \theta_j \sum_{x \in \mathbb{Z}} N_{[nt_j]}(x) \zeta_j(x)$$

with $\zeta_j(x) = \mathbb{1}_{\{\xi_x \le s_j\}} - s_j$

Sketch of Proof III

- condition on $(S_n)_{n\in\mathbb{N}}$ and use conditional independence

$$\varphi_n(\lambda) := E\Big(\exp\big(i\lambda n^{-1+\frac{1}{2\alpha}}\sum_{j=1}^{k}\theta_j\sum_{x\in\mathbb{Z}}N_{[nt_j]}(x)\zeta_j(x)\big)\Big)$$

$$= E\Big(\prod_{x\in\mathbb{Z}}E\Big(\exp\big(i\lambda n^{-1+\frac{1}{2\alpha}}\sum_{j=1}^{k}\theta_j N_{[nt_j]}(x)\zeta_j(x)\big)\Big|(X_n)_{n\in\mathbb{N}}\Big)\Big)$$

- finite demensional convergence: Taylor expansion and Lévy's continuity theorem
- tightness: fourth moment inequality and lemma by Bickel and Wichura (1971)

Properties of Limit Process

- self-similar with exponent $1 - \frac{1}{2\alpha}$
- for fixed s: $(W(s,t))_{t\in[0,1]}$ special case of process Δ
- for fixed t: conditional on random walk S_n Brownian bridge, so $(W(s,t))_{s\in[0,1]}$ mixture of Brownian bridges
- thus rough paths
- marginal distributions: not Gaussian (Gaussian mixtures)
- after continuous modification: for any $\gamma < 1 - \frac{1}{2\alpha}$, $\gamma' < \frac{1}{2}$ there exists almost surely finite $C_{\gamma,\gamma'}$, such that

$$\left|W(s,t) - W(s',t')\right| \leq C_{\gamma,\gamma'} \left(|s-s'|^{\gamma'} + |t-t'|^{\gamma}\right).$$

Continuity of Limit Process

nonuniform Kolmogorov-Chentsov theorem

Lemma

$(\mathbf{X}_t)_{t\in[0,1]^d}$ *stochastic process, for some* $m \geq 1$, $c_1, \ldots, c_d, \beta_1, \ldots, \beta_d$:

$$E\left[|\mathbf{X}_t - \mathbf{X}_s|^m\right] \leq \sum_{i=1}^{d} c_i \, |t_i - s_i|^{d+\beta_i} \,.$$

for all $t = (t_1, \ldots, t_d)$, $s = (s_1, \ldots, s_d)$, *then for* $\gamma_1, \ldots, \gamma_d$ *with* $\gamma_i < \frac{\beta_i}{m}$ *there exists a continuous modification* $\tilde{\mathbf{X}}$ *with*

$$\left|\tilde{\mathbf{X}}_t - \tilde{\mathbf{X}}_s\right| \leq C_{\gamma_1,\ldots,\gamma_d} \sum_{i=1}^{d} |t_i - s_i|^{\gamma_i} \,.$$

Comparision

	independent	random walk in random scenery	lrd Gaussian
exponent of self similarity	$1/2$	$1-\frac{1}{2\alpha} > 1/2$	$\frac{2-mD}{2} > 1/2$
marginal distr.	Gaussian	not Gaussian: Gaussian mixture	not Gaussian if $m > 1$
γ_t-Hölder-cont. t direction	$\gamma_t < 1/2$	$\gamma_t > 1/2$	$\gamma_t > 1/2$
γ_s-Hölder-cont. s direction	$\gamma_s < 1/2$	$\gamma_s < 1/2$	$\gamma_s = 1$

Definition

- $(X_n)_{n\in\mathbb{N}}$ stationary sequence of random variables
- kernel: $h : \mathbb{R}^2 \to \mathbb{R}$ measurable and symmetric

Definition

U-statistic:

$$U_n(h) := \sum_{1\leq i<j\leq n} h(X_i, X_j)$$

von Mises-statistic:

$$V_n(h) := \sum_{i,j=1}^{n} h(X_i, X_j)$$

Relation to Empirical Process

- empirical distribution function: $F_n(s) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X_i \le t}$
- without loss of generality: distribution function $F(s) = s$
- under independence: $E[h(X_i, X_j)] = \iint h(x,y)dF(x)dF(y)$, so $E[V_n] \approx n^2 \iint h(x,y)dF(x)dF(y)$

$$
\begin{aligned}
V_n - n^2 \iint h(x,y)dF(x)dF(y) \\
= n^2 \iint h(x,y)dF_n(x)dF_n(y) - n^2 \iint h(x,y)dF(x)dF(y) \\
= 2n \iint h(x,y)dF(x)dW_n(y,1) + \iint h(x,y)dW_n(x,1)dW_n(y,1)
\end{aligned}
$$

Linear Part

- set $h_1(x) := \int h(x,y)dF(y) = E[h(x,X_1)]$
- assume: h_1 of bounded variation

$$2n \iint h(x,y)dF(x)dW_n(y,1) = 2n \int h_1(y)dW_n(y,1)$$
$$= 2n \int W_n(s,1)dh_1(s)$$

- by continuous mapping theorem

$$n^{-2+\frac{1}{2\alpha}} 2n \iint h(x,y)dF(x)dW_n(y,1)$$
$$= 2\int n^{-1+\frac{1}{2\alpha}} W_n(s,1)dh_1(s) \Rightarrow 2\int W(s,1)dh_1(s)$$

Degenerate Part

- note that $\|W_n\|_\infty = O_p(n^{1-\frac{1}{2\alpha}})$
- assumption: signed measure induced by h of bounded total variation

$$\iint h(x,y)dW_n(x,1)dW_n(y,1) = O(\|W_n\|_\infty^2) = O_p(n^{2-\frac{1}{\alpha}})$$
$$= o_P(n^{2-\frac{1}{2\alpha}})$$

- by Slutsky's theorem

$$n^{-2+\frac{1}{2\alpha}}\left(V_n - n^2 \iint h(x,y)dF(x)dF(y)\right)$$
$$= n^{-1+\frac{1}{2\alpha}} 2\int W_n(s,1)dh_1(s) + n^{-2+\frac{1}{2\alpha}} \iint h(x,y)dW_n(x,1)dW_n(y,1)$$
$$\Rightarrow 2\int W(s,1)dh_1(s)$$

Examples

- $h(x, y) = |x - y|$

$$\frac{2}{n(n-1)} U_n(h) = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |X_i - X_j|,$$

Gini's mean difference

- $h(x, y) = \frac{1}{2}(x - y)^2$

$$\frac{2}{n(n-1)} U_n(h) = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

sample variance

Conditions on kernel

- conditions on total variation: imply boundedness of h
- examples: h not bounded
- Guillotin-Plantard, Ladret (2005): fourth moments
- Franke, Pène, Wendler (2016+): domain of β-stable law for $\beta \in (1, 2]$
- Franke, Pène, Wendler (2017): domain of β-stable for $\beta \in (0, 2]$

Case $\beta \in (1, 2]$

Assumptions:

- $h_1(\xi(1))$ in normal domain of attraction of β-stable law, $1 < \beta \leq 2$
- $E|h(\xi(1), \xi(2))|^\eta < \infty$ with $\eta = \frac{2\beta'}{1+\beta'}$ for a $\beta' > \beta$
- $E[h(\xi(1), \xi(2))] = 0$

Theorem (Franke, Pène, W., 2016+)

If $1 < \alpha \leq 2$*, then in* $C[0, 1]$

$$\left(n^{-2+\frac{1}{\alpha}-\frac{1}{\alpha\beta}} U_n(t)\right)_{t\in[0,1]} \Rightarrow (\Delta_t)_{t\in[0,1]},$$

- $U_n(t) = U_k$ if $nt = k$, linear interpolated in between
- $\Delta_t = \int L_t(x) dZ(x)$

Hoeffding decomposition

$$U_n = (n-1)\sum_{i=1}^{n} h_1(\xi(S_i)) + \sum_{1\le i<j\le n} h_2(\xi(S_i),\xi(S_j))$$

with $h_1(x) := E(h(x,\xi(1)))$
$h_2(x,y) := h(x,y) - h_1(x) - h_1(y).$

Lemma

$$\max_{k\le n} \sum_{1\le i<j\le k} h_2(\xi(S_i),\xi(S_j)) = o(n^{2-\frac{1}{\alpha}+\frac{1}{\alpha\beta}})$$

almost surely

Case $\beta \in (0, 2)$

Assumptions:

- $h(\xi_1, \xi_2)$ in domain of β-stable law with $\beta \in (0, 2)$
- there exist $C_0 > 0$ and $\gamma > \frac{3\beta}{4}$ such that for all $z, z' \in (1, +\infty)$

$$P\left(|h(\xi_1, \xi_2)| \geq z \text{ and } |h(\xi_1, \xi_3)| \geq z'\right) \leq C_0 \left(zz'\right)^{-\gamma}$$

- if $\beta > 1$, then $\mathbb{E}[h(\xi_1, \xi_2)] = 0$
- if $\beta \geq 4/3$, there exists $C_0' > 0$ and $\theta' > \frac{3\beta}{4} - 1$ such that

$$\forall M, M' \in (0, +\infty), \quad |\mathbb{E}\left[\mathbf{h}_M(\xi_1, \xi_2)\mathbf{h}_{M'}(\xi_1, \xi_3)\right]| \leq C_0'(MM')^{-\theta'},$$

  where $\mathbf{h}_M(x, y) := h(x, y)\mathbf{1}_{\{|h(x,y)| \leq M\}} + \frac{\beta}{\beta - 1}(c_0 - c_1)M^{1-\beta}$
- if $\beta = 1$, then $\lim_{M \to +\infty} \mathbb{E}[h(\xi_1, \xi_2)\mathbf{1}_{\{|h(\xi_1, \xi_2)| \leq M\}}] = 0$

Weak Convergence

Theorem (Franke, Pène, W., 2017)

$$\left(n^{-2+\frac{2}{\alpha}-\frac{2}{\alpha\beta}} U_{\lfloor nt \rfloor}\right)_{t\in[0,T]} \Rightarrow \left(\int_{\mathbb{R}^2} L_t(x) L_t(y)\, dZ_{x,y}\right)_{t\in[0,T]}$$

in the Skorokhod space $D([0,T])$ endowed with the J_1-metric

- $(Z_{x,y})_{x,y\in\mathbb{R}}$: β-stable Lévy-sheet
- method of proof: point processes
- for for $\beta \in (1,2]$: rate faster than in previous theorem
- condtions imply: $h_1 = 0$, thus linear part vanishes

Conclusion

- functional non-central limit theorem for sequential empirical process
- some properties like in independent case (roughness of the paths), some properties like in long memory case (not Gaussian, self-similarity)
- application: *U*-statistics (restrictive conditions)
- less restrictive conditions: other methods like Hoeffding-decomposition or point processes

Thank you for your attention!

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