Good and Bad News for the Block Bootstrap of (Generalized) Quantiles

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joined work with O.Sh. Sharipov

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- $(X_n)_{n\in\mathbb{N}}$ stationary process, real valued
- marginal distribution function F with $F(t) = P(X_i \leq t)$
- $p \in (0,1)$

Definition

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is called p-quantile.

- empirical distribution function $F_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq t\}}$
- F_n consistent estimator of F

Definition

$$F_n^{-1}(p) := \inf\left\{t \middle| F_n(t) \geq p\right\}$$

is called empirical p-quantile.

$(X_n)_{n\in\mathbb{N}}$ iid

Theorem

F differentiable in $F^{-1}(p)$ and $f(F^{-1}(p)) = F(F^{-1}(p)) > 0$, then

$$\sqrt{n}\left(F_n^{-1}(p) - F^{-1}(p)\right) \xrightarrow{\mathcal{D}} N(0, \frac{p(1-p)}{f^2(F^{-1}(p))})$$

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proof:

- central limit theorem for $F_n(F^{-1}(p))$
- Bahadur-Ghosh representation (Ghosh 1971)

$$F_n^{-1}(p) - F^{-1}(p) + \frac{F_n(F^{-1}(p)) - p}{f(F^{-1}(p))} = o_P(\frac{1}{\sqrt{n}})$$

- $f'(F^{-1}(p))$ differentiable is necessary for CLT (de Haan, Taconis-Haantjes 1979)
- nonnormal limit otherwise

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Theorem (Haan, Taconis-Haantjes 1979)

If

$$F(F^{-1}(p)+h)-F(F^{-1}(p)) = M|h|^{\rho}\,\mathrm{sgn}(h)+o(|h|^{\rho})$$

then

$$F_n^{-1}(p)-F^{-1}(p) = \left(\frac{|p-F_n\left(F^{-1}(p)\right)|}{M}\right)^{\frac{1}{\rho}}\mathrm{sgn}\left(p-F_n(F^{-1}(p))\right)+R_n,$$

where $R_n = o_P(n^{-\frac{1}{2\rho}})$.

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$$n^{\frac{1}{2\rho}}(F_n^{-1}(p)-F^{-1}(p)) \to C|W|^{\frac{1}{\rho}}\,\mathrm{sgn}(W)$$

in distribution, where C is a constant and W is a normal random variable.

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- $C|W|^{\frac{1}{\rho}}\operatorname{sgn}(W)$ not normal for $\rho \neq 1$
- question: similar results for dependent data?

Let $(X_n)_{n\in\mathbb{N}}$ be a stationary process.

Definition

The strong mixing coefficient is given by

$$\alpha(k) = \sup_{n\in\mathbb{N}} \sup_{A\in\mathcal{F}_1^n, B\in\mathcal{F}_{n+k}^\infty} |P(A\cap B) - P(A)P(B)|,$$

where $\mathcal{F}_b^a$ is the σ-field generated by r.v.'s $X_a, \dots, X_b$. $(X_n)_{n\in\mathbb{N}}$ is called *strongly mixing*, if $\alpha(k) \to 0$ as $k \to \infty$.

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examples:

- ARMA-processes (under additional assumptions on the innovations)
- GARCH-processes (under additional assumptions on the innovations)

Theorem

If $\sum_{n=1}^{\infty} \alpha(n) < \infty$ *and*

$$F(F^{-1}(p) + h) - F(F^{-1}(p)) = M|h|^{\rho} \operatorname{sgn}(h) + o(|h|^{\rho})$$

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$$F_n^{-1}(p) - F^{-1}(p) = \left(\frac{|p - F_n\left(F^{-1}(p)\right)|}{M} \right)^{\frac{1}{\rho}} \operatorname{sgn}(p - F_n\left(F^{-1}(p)\right)) + R_n,$$

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main ingredients of proof:

- Ibragimov (1962): CLT for $\sqrt{n}(F_n(F^{-1}(p)) - p)$
- $\sqrt{n}(F_n(F^{-1}(p)) - F_n(F^{-1}(p_n')) - (p - p_n')) \to 0$ in probability if $p_n' - p \to 0$

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$$n^{\frac{1}{2\rho}}(F_n^{-1}(p) - F^{-1}(p)) \to C|W|^{\frac{1}{\rho}}\,\text{sgn}(W)$$

in distribution, where C *is a constant and* W *is a normal random variable.*

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- for $\rho = 1$:$C|W|^{\frac{1}{\rho}} \operatorname{sgn}(W) = CW$ normal
- Sun, Lahiri (2006): $\rho = 1$, $\alpha(n) = O(n^{-1-\epsilon})$
- for $\rho \neq 1$ nonnormal limit

circular block bootstrap:

- choose block length $l = l_n$ with $\frac{1}{l_n} + \frac{l_n}{n} \to 0$
- extend the sample $X_1, \ldots, X_n$ periodically by $X_{i+n} = X_i$
- construct a new sample by drawing blocks of length $p = p(n)$ with replacement $k = \left[\frac{n}{p}\right]$ times, so for $i = 1, 2, \ldots, k$, $j = 1, \ldots, n$:

$$P\left[\left(X^\star_{(i-1)p+1}, \ldots, X^\star_{ip}\right) = (X_j, \ldots, X_{j+l-1})\right] = \frac{1}{n}$$

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bootstrap estimator for empirical distribution function:

$$F^\star_n(t) := \frac{1}{\sqrt{kl_n}} \sum_{i=1}^{kl_n} \mathbb{1}_{\{X^\star_i \leq t\}}$$

Theorem (Radulović (1996))

If X_i is bounded and $\sum_{n=1}^{\infty} \alpha(n) < \infty$

$$\sup_{x\in\mathbb{R}} \left| P^* \left(\frac{1}{\sqrt{kp}} (\sum_{i=1}^{kp} X_i^* - \sum_{i=1}^{kp} X_i) \leq x \right) - P \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - EX_1) \leq x \right) \right| \to 0$$

in probability.

Theorem

If $\sum_{n=1}^{\infty} \alpha(n) < \infty$ *and F differentiable in* $F^{-1}(p)$*, then*

$$F_n^{\star -1}(p) - F^{-1}(p) = \frac{p - F_n^{\star}(F^{-1}(p))}{f(F^{-1}(p))} + R_n^{\star},$$

where $R_n^{\star} = o_P(n^{-\frac{1}{2}})$.

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where $R_n^{\star} = o_P(n^{-\frac{1}{2}})$.

main ingredients of proof:

- $\sqrt{n}(F_n^{\star}(F^{-1}(p)) - p) = \sqrt{n}(F_n^{\star}(F^{-1}(p)) - F_n(F^{-1}(p))) + \sqrt{n}(F_n(F^{-1}(p)) - p)$ bounded in probability
- $\sqrt{n}(F_n^{\star}(F^{-1}(p)) - F_n^{\star}(F^{-1}(p_n')) - (p - p_n')) \to 0$ in probability if $p_n' \to 0$

$$
\begin{aligned}
F_n^{\star-1}(p) - F_n^{-1}(p) &= \Big(F_n^{\star-1}(p) - F^{-1}(p)\Big) - \Big(F_n^{-1}(p) - F^{-1}(p)\Big) \\
&= \frac{p - F_n^{\star}(F^{-1}(p))}{f(F^{-1}(p))} - \frac{p - F_n(F^{-1}(p))}{f(F^{-1}(p))} + R_n^{\star} - R_n \\
&= \frac{F_n(F^{-1}(p)) - F_n^{\star}(F^{-1}(p))}{f(F^{-1}(p))} + R_n^{\star} - R_n
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with $R_n^{\star} - R_n = o_P(n^{-\frac{1}{2}})$

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$$= \frac{F_n(F^{-1}(p)) - F_n^\star(F^{-1}(p))}{f(F^{-1}(p))} + R_n^\star - R_n$$

with $R_n^\star - R_n = o_P(n^{-\frac{1}{2}})$

Theorem

$$\sup_{t\in\mathbb{R}} \left| P^\star\left(F_n^{\star -1}(p) - F_n^{-1}(p) \le t\right) - P\left(F_n^{-1}(p) - F^{-1}(p) \le t\right)\right| \xrightarrow{n\to\infty} 0$$

in probability.

Theorem

If $\alpha(n) = O(n^{-1-\epsilon})$*, F differentiable in* $F^{-1}(p)$ *and*

$$p(n) = p(2^l) \text{ for } 2^l < n \leq 2^{l+1}, \quad p(n) \leq Cn^{1-\epsilon} \text{ for some } \epsilon > 0$$

then

$$\sup_{t \in \mathbb{R}} \left| P^\star\left(F_n^{\star-1}(p) - F_n^{-1}(p) \leq t\right) - P\left(F_n^{-1}(p) - F^{-1}(p) \leq t\right)\right| \xrightarrow{n\to\infty} 0$$

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Sun, Lahiri (2006): $\alpha(n) = O(n^{-9.5-\epsilon})$, $l_n = O(n^{0.5-\eta})$ for some $\eta > \frac{5}{40+4\epsilon}$

Theorem

If $\sum_{n=1}^{\infty} \alpha(n) < \infty$ *and*

$$F(F^{-1}(p) + h) - F(F^{-1}(p)) = M|h|^{\rho} \operatorname{sgn}(h) + o(|h|^{\rho})$$

then

$$F_n^{\star-1}(p) - F^{-1}(p) = \left(\frac{|p - F_n^{\star}(F^{-1}(p))|}{M} \right)^{\frac{1}{\rho}} \operatorname{sgn}\left(p - F_n^{\star}(F^{-1}(p))\right) + R_n^{\star},$$

where $R_n^{\star} = o_P(n^{-\frac{1}{2\rho}})$.

with $g(x) = (\frac{|x|}{M})^{\frac{1}{\rho}} \operatorname{sgn}(x)$

$$g\left(F_n\left(F^{-1}(p)\right) - F_n^{\star}\left(F^{-1}(p)\right)\right) \neq g\left(p - F_n^{\star}\left(F^{-1}(p)\right)\right) + g\left(p - F_n\left(F^{-1}(p)\right)\right)$$

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Theorem

Under assumptions of theorem above

$$\sup_{t\in\mathbb{R}} \left| P^\star\left(F_n^{\star-1}(p) - F_n^{-1}(p) \leq t\right) - P\left(F_n^{-1}(p) - t_p \leq t\right)\right| \xrightarrow{n\to\infty} Z_\rho$$

in distribution, where Z_ρ is a non-degenerate (non-constant) random variable.

$h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ symmetric, measurable function

Definition (Empirical *U*-distribution function)

We define

$$U_n(t) = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \mathbb{1}_{\{h(X_i, X_j) \leq t\}}.$$

$(U_n(t))_{t \in \mathbb{R}}$ is called empirical *U*-distribution function.

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$(U_n(t))_{t \in \mathbb{R}}$ is called empirical *U*-distribution function.

- generalization of empirical distribution function
- natural estimator for *U*-distribution function $U(t) = P[h(X, Y) \leq t]$, where X, Y independent

Definition (U-Quantile)

Let be $p \in (0, 1)$.

$$U_n^{-1}(p) := \inf \left\{ t \middle| U_n(t) \geq p \right\}$$

called the p-th empirical U-quantile.

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- natural estimator of p-th U-quantile $U^{-1}(p)$
- applications in robust statistics

Hodges-Lehmann estimator for location:
$h(x, y, t) = \mathbb{1}_{\{(x+y/2)\leq t\}}$, $p = \frac{1}{2}$

$$U_n^{-1}(p) = \text{median}\left\{\frac{X_i + X_j}{2}\middle| 1 \leq i < j \leq n\right\}$$

- 96% asymptotic efficiency in i.i.d. normal model
- 29% breakdown point

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median of absolute differences estimator of scale:
$h(x, y, t) = \mathbb{1}_{\{|x-y|\leq t\}}$, $p = \frac{1}{2}$

$$U_n^{-1}(p) = 1.0483\,\text{median}\left\{ |X_i - X_j| \middle| 1 \leq i < j \leq n \right\}$$

- 86% asymptotic efficiency in i.i.d. normal model
- 29% breakdown point

Definition

The absolute regularity coefficient is given by

$$\beta(k) = \sup_{n\in\mathbb{N}} E \sup\{\left|P(A|\mathcal{F}_{-\infty}^{n}) - P(A)\right| : A \in \mathcal{F}_{n+k}^{\infty}\}$$

and $(X_n)_{n\in\mathbb{N}}$ is called *absolutely regular*, if $\beta(k) \to 0$ as $k \to \infty$.

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and $(X_n)_{n\in\mathbb{N}}$ is called *absolutely regular*, if $\beta(k) \to 0$ as $k \to \infty$.

- absolute regularity (β mixing) implies strong mixing (α mixing)
- allows for coupling methods: replacement of dependent random variables by independent

plug in estimator

$$U_n^\star(t) := \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} \mathbb{1}_{\left\{h\left(X_i^\star, X_j^\star\right)\leq t\right\}}$$

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Theorem

If $(X_n)_{n\in\mathbb{N}}$ is absolutely regular with mixing coefficients $\sum_{i=1}^{\infty} \beta(n) < \infty$, then for fixed t

$$\sup_{x\in\mathbb{R}} \Bigg| P^\star\left[\sqrt{pk}\left(U_n^\star(t) - E^\star\left[U_n^\star(t)\right]\right) \leq x\right] - P\left[\sqrt{n}\left(U_n(t) - U(t)\right) \leq x\right]\Bigg| \to 0$$

in probability.

Theorem

If $\sum_{n=1}^{\infty} \beta(n) < \infty$ *and* U *differentiable in* $U^{-1}(p)$*, then*

$$U_n^{-1}(p) - U^{-1}(p) = \frac{p - U_n(U^{-1}(p))}{U'(U^{-1}(p))} + R_n,$$

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$$\sup_{x\in\mathbb{R}}\left|P^{\star}\left[\sqrt{pk}\left(U_n^{\star-1}(t)-E^{\star}\left[U_n^{\star-1}(t)\right]\right)\leq x\right]\right.$$
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$$\left.-P\left[\sqrt{n}\left(U_n^{-1}(t)-U^{-1}(t)\right)\leq x\right]\right|\to 0$$

in probability.

Theorem holds also

- under strong mixing with additional continuity condition
- with almost sure convergence under additional conditions on mixing rate and block length

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Thank you for your attention!

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